

# A CONGRUENCE INVOLVING THE QUOTIENTS OF EULER AND ITS APPLICATIONS (III)

HAO ZHONG, SHANE CHERN, AND TIANXIN CAI

ABSTRACT. In this paper, we will present several new congruences involving binomial coefficients under integer moduli, which are the continuation of the previous two work by Cai *et al.* (2002, 2007).

## 1. INTRODUCTION

In 1895, Morley [5] proved the following beautiful and profound congruence involving binomial coefficients, that is, for any prime  $p \geq 5$ ,

$$(-1)^{(p-1)/2} \binom{p-1}{(p-1)/2} \equiv 4^{p-1} \pmod{p^3}. \quad (1.1)$$

However, his proof, which is due to an explicit form of De Moivre's theorem, fails to deal with other binomial coefficients. In 2002, Cai [1] extended Morley's congruence to integer moduli through a generalization of Lehmer's congruence. More precisely, he proved

$$\prod_{d|n} \binom{d-1}{(d-1)/2}^{\mu(n/d)} \equiv (-1)^{\phi(n)/2} 4^{\phi(n)} \begin{cases} \pmod{n^3} & \text{if } 3 \nmid n, \\ \pmod{n^3/3} & \text{if } 3 \mid n, \end{cases} \quad (1.2)$$

for odd  $n > 1$ . When  $n$  is an odd prime  $p \geq 5$ , (1.2) becomes (1.1). According to Cosgrave and Dilcher [3], (1.2) appears to be the first analogue for composite moduli of a ‘‘Lehmer type’’ congruence. Later in 2007, Cai *et al.* [2] proposed several new congruences of the same type, in which  $(d-1)/2$  is replaced by  $\lfloor d/3 \rfloor$ ,  $\lfloor d/4 \rfloor$ , and  $\lfloor d/6 \rfloor$ , respectively, where  $\lfloor x \rfloor$  denotes the largest integer not greater than  $x$ . In this paper, we will further extend the work in [1] and [2].

First, we introduce a new generalization of Euler's totient. For  $k$  an integer and  $f$  a number theoretic function, we define

$$\phi_f^{(k)}(n) := \sum_{d|n} \left(\frac{n}{d}\right)^k f(d) \mu(d). \quad (1.3)$$

And if  $f \equiv 1$ , then  $\phi_f^{(k)}(n)$  equals to Jordan totient function. It's easy to prove that  $\phi_f^{(k)}(n) = n^k \prod_{p|n} (1 - f(p)p^{-k})$  when  $f(n)$  is multiplicative. And Our main results are

**Theorem 1.1.** *Let  $n$  be a positive integer and  $(n, 6) = 1$ . For  $e = 2, 3, 4$  or  $6$ , we have*

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$$\sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \frac{1}{r^2} \equiv -J_e(n) n^{\phi(n)-2} \phi_{J_e}^{(2-\phi(n))}(n) \frac{B_{\phi(n)-1}(\frac{1}{e})}{\phi(n)-1} \pmod{n}, \quad (1.4)$$

where  $B_n$  is the  $n$ th Bernoulli number,  $B_n(x)$  is the Bernoulli polynomial, and  $J_e(n)$  is the Jacobi symbol for  $n$  and  $e$ ,

$$J_e(n) = \left(\frac{n}{e}\right) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{e} \\ -1 & \text{if } n \equiv -1 \pmod{e} \end{cases}$$

since  $(n, 6) = 1$ .

According to Lehmer [4], for  $v$  odd,  $B_v(\frac{1}{2}) = 0$  and  $B_v(\frac{1}{4}) = -\frac{vE_{v-1}}{4^v}$  where  $E_m$  is the  $m$ th Euler number defined by the generating function

$$\frac{1}{\cosh x} = \sum_{m=0}^{\infty} \frac{E_m}{m!} \cdot x^m.$$

It follows

**Corollary 1.1.** *For  $n$  integer and  $(n, 6)=1$ ,*

$$\sum_{\substack{r=1 \\ (r,n)=1}}^{\frac{n-1}{2}} \frac{1}{r^2} \equiv 0 \pmod{n}. \quad (1.5)$$

**Corollary 1.2.** *For  $n$  integer and  $(n, 6)=1$ ,*

$$\sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/4 \rfloor} \frac{1}{r^2} \equiv (-1)^{\frac{n-1}{2}} 4n^{\phi(n)-2} \phi_{J_4}^{(2-\phi(n))}(n) E_{\phi(n)-2} \pmod{n} \quad (1.6)$$

**Corollary 1.3.** *For  $n$  a positive integer, we have*

(1) *if  $(n, 6) = 1$ ,*

$$\sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/3 \rfloor} \frac{1}{r} \equiv -\frac{3}{2}q_3(n) + \frac{3}{4}nq_3^2(n) + \frac{1}{3}J_3(n)n^{\phi(n)-1} \phi_{J_3}^{(2-\phi(n))}(n) \frac{B_{\phi(n)-1}(\frac{1}{3})}{\phi(n)-1} \pmod{n^2}; \quad (1.7)$$

(2) *if  $(n, 6) = 1$ ,*

$$\sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/4 \rfloor} \frac{1}{r} \equiv -3q_2(n) + \frac{3}{2}nq_2^2(n) + (-1)^{\frac{n+1}{2}} n^{\phi(n)-1} \phi_{J_4}^{(2-\phi(n))}(n) E_{\phi(n)-2} \pmod{n^2}; \quad (1.8)$$

(3) *if  $(n, 30) = 1$ ,*

$$\sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/6 \rfloor} \frac{1}{r} \equiv -2q_2(n) - \frac{3}{2}q_3(n) + nq_2^2(n) + \frac{3}{4}nq_3^2(n) + \frac{1}{6}J_6(n)n^{\phi(n)-1}\phi_{J_6}^{(2-\phi(n))}(n) \frac{B_{\phi(n)-1}(\frac{1}{6})}{\phi(n)-1} \pmod{n^2}. \quad (1.9)$$

**Theorem 1.2.** *For any positive integer  $k$  and odd  $n > 1$ , it follows*

$$\prod_{d|n} \binom{kd-1}{(d-1)/2}^{\mu(n/d)} \equiv (-1)^{\phi(n)/2} 4^{k\phi(n)} \begin{cases} \pmod{n^3} & \text{if } 3 \nmid n, \\ \pmod{n^3/3} & \text{if } 3 \mid n. \end{cases} \quad (1.10)$$

**Remarks.** (1). One immediately sees that (1.10) gives a generalization of Morley's congruence (1.1) if we let  $n$  be an odd prime  $p \geq 5$ .

**Corollary 1.4.** *Let  $p \geq 5$  be an odd prime. For any positive integer  $k$ , it follows*

$$(-1)^{(p-1)/2} \binom{kp-1}{(p-1)/2} \equiv 4^{k(p-1)} \pmod{p^3}. \quad (1.11)$$

(2). If we let  $n$  be the product of two distinct odd primes, we obtain the following result resembling the quadratic reciprocity law, which extends Cai's Corollary 4 in [1].

**Corollary 1.5.** *Let  $p, q$  be two distinct odd primes. For any positive integer  $k$ , it follows*

$$\binom{kpq-1}{(pq-1)/2} \equiv 4^{k(p-1)(q-1)} \binom{kp-1}{(p-1)/2} \binom{kq-1}{(q-1)/2} \pmod{p^3q^3}. \quad (1.12)$$

In order to make the the following theorem briefly, we set

$$A_e(n) := J_e(n)n^{\phi(n)-2}\phi_{J_e}^{(2-\phi(n))}(n) \frac{B_{\phi(n)-1}(\frac{1}{e})}{\phi(n)-1}.$$

Then, we have

**Theorem 1.3.** *For any positive integer  $k$  and odd  $n > 1$ , it follows*

(1) *if  $(3, n) = 1$ ,*

$$\prod_{d|n} \binom{kd-1}{\lfloor d/3 \rfloor}^{\mu(n/d)} \equiv (-1)^{\phi_3(n)} \left\{ \frac{1}{2}(27^{k\phi(n)} + 1) + k\left(\frac{1}{2}k - \frac{1}{3}\right)n^2 A_3(n) \right\} \pmod{n^3}; \quad (1.13)$$

(2) *if  $(3, n) = 1$ ,*

$$\prod_{d|n} \binom{kd-1}{\lfloor d/4 \rfloor}^{\mu(n/d)} \equiv (-1)^{\phi_4(n)} \left\{ 8^{k\phi(n)} + (-1)^{\frac{n+1}{2}} k(2k-1)n^{\phi(n)} \phi_{J_4}^{(2-\phi(n))}(n) E_{\phi(n)-2} \right\} \pmod{n^3}; \quad (1.14)$$

(3) if  $(15, n) = 1$ ,

$$\prod_{d|n} \binom{kd-1}{\lfloor d/6 \rfloor}^{\mu(n/d)} \equiv (-1)^{\phi_6(n)} \left\{ \frac{1}{2}(16^{k\phi(n)} + 27^{k\phi(n)}) + \frac{1}{2}k(k - \frac{1}{3})n^2 A_6(n) \right\} \pmod{n^3}. \quad (1.15)$$

Here

$$\phi_e(n) := \sum_{d|n} \mu\left(\frac{n}{d}\right) \left\lfloor \frac{d}{e} \right\rfloor$$

is the generalized Euler totient function defined in [2].

Furthermore, we consider the following generalized binomial coefficient. For  $x \in \mathbb{C}$ , let

$$\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n(n-1)\cdots 1},$$

if  $n$  is a positive integer, and  $\binom{x}{0} = 1$ . If we replace  $kd-1$  in Theorem 1.2 by  $(kd-1)/2$ , then we have a new congruence.

**Theorem 1.4.** *For positive integer  $k$  and odd  $n > 1$ , it follows*

$$\prod_{d|n} \binom{(kd-1)/2}{(d-1)/2}^{\mu(n/d)} \equiv 2^{-(k-1)\phi(n)} \begin{cases} \pmod{n^3} & \text{if } 3 \nmid n, \\ \pmod{n^3/3} & \text{if } 3 \mid n. \end{cases} \quad (1.16)$$

## 2. PRELIMINARIES

To prove the theorems, we need the following lemmas.

**Lemma 2.1** ([2, Lemma 1]). *If  $p \geq 5$  is a prime and  $k \geq 2$ ,  $l, t$  are positive integers, and  $s$  is the least positive residue of  $p^l$  modulo  $t$ , then*

$$\sum_{r=1}^{\lfloor p^l/t \rfloor} (p^l - tr)^{2k} \equiv \frac{t^{2k}}{2k+1} \left\{ \frac{2k+1}{t} p^l B_{2k} - B_{2k+1}\left(\frac{s}{t}\right) \right\} \pmod{p^{3l-1}}, \quad (2.1)$$

where  $B_n$  is the  $n$ th Bernoulli number.

**Lemma 2.2** ([1, Lemma 1]). *Let  $n > 1$  be an integer. Then*

$$\sum_{\substack{i=1 \\ (i,n)=1}}^{n-1} \frac{1}{i^2} \equiv 0 \begin{cases} \pmod{n} & \text{if } 3 \nmid n, n \neq 2^a, \\ \pmod{n/3} & \text{if } 3 \mid n, \\ \pmod{n/2} & \text{if } n = 2^a. \end{cases} \quad (2.2)$$

**Lemma 2.3** ([6, Corollary 1.3]). *Let  $a \in \mathbb{Z}$ ,  $k, q, m \in \mathbb{Z}^+$  and  $(m, q) = 1$ . Then*

$$\frac{1}{k} (m^k B_k(\frac{x+a}{m}) - B_k(x)) \equiv \sum_{j=0}^{q-1} \left( \left\lfloor \frac{a+jm}{q} \right\rfloor + \frac{1-m}{2} \right) (x+a+jm)^{k-1} \pmod{q}. \quad (2.3)$$

**Lemma 2.4** ([1, Theorem 1]). *For odd  $n > 1$ , we have*

$$\sum_{\substack{i=1 \\ (i,n)=1}}^{(n-1)/2} \frac{1}{i} \equiv -2q_2(n) + nq_2^2(n) \pmod{n^2}, \quad (2.4)$$

where  $q_r(n)$  denotes the Euler quotient, i.e.,

$$q_r(n) = \frac{r^{\phi(n)} - 1}{n},$$

with  $(n, r) = 1$ .

**Lemma 2.5** ([2, Theorem 1]). *For odd  $n > 1$ , we have*

(1) *if  $(3, n) = 1$ ,*

$$\sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor d/3 \rfloor} \frac{1}{n-3r} \equiv \frac{1}{2}q_3(n) - \frac{1}{4}nq_3^2(n) \pmod{n^2}; \quad (2.5)$$

(2) *if  $(3, n) = 1$ ,*

$$\sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor d/4 \rfloor} \frac{1}{n-4r} \equiv \frac{3}{4}q_2(n) - \frac{3}{8}nq_2^2(n) \pmod{n^2}; \quad (2.6)$$

(3) *if  $(15, n) = 1$ ,*

$$\sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor d/6 \rfloor} \frac{1}{n-6r} \equiv \frac{1}{3}q_2(n) + \frac{1}{4}q_3(n) - \frac{1}{6}nq_2^2(n) - \frac{1}{8}nq_3^2(n) \pmod{n^2}. \quad (2.7)$$

### 3. PROOFS OF THE THEOREMS

*Proof of Theorem 1.1.* First of all, we prove that for  $p$  prime and  $l$  positive integer,

$$\sum_{\substack{r=1 \\ p \nmid r}}^{\lfloor p^l/e \rfloor} \frac{1}{r^2} \equiv -J_e(p^l) \frac{B_{\phi(p^l)-1}(\frac{1}{e})}{\phi(p^l)-1} \pmod{p^l}. \quad (3.1)$$

By using the multiplication theorems given by Joseph Ludwig Raabe in 1851: for natural number  $m \geq 1$ ,  $B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n(x + \frac{k}{m})$ , we can obtain the values of  $B_n(\frac{s}{t})$ . Fix  $x$  zero and  $n$  odd,

- if  $m = 2$ , then  $B_n(\frac{1}{2}) = 0$ ;
- if  $m = 3$ , then  $B_n(\frac{1}{3}) = -B_n(\frac{2}{3})$ ;
- if  $m = 4$ , then  $B_n(\frac{1}{4}) = -B_n(\frac{3}{4})$ ;
- if  $m = 6$ , then  $B_n(\frac{1}{6}) = -B_n(\frac{5}{6})$ .

Taking  $2k = \phi(p^l) - 2$  and  $t = e$  in (2.1) and using the von Staudt-Clauson theorem, we have

$$\begin{aligned} \sum_{\substack{r=1 \\ p \nmid r}}^{\lfloor p^l/e \rfloor} \frac{1}{r^2} &\equiv \sum_{\substack{r=1 \\ p \nmid r}}^{\lfloor p^l/e \rfloor} \frac{e^2}{(p^l - er)^2} \equiv \sum_{r=1}^{\lfloor p^l/e \rfloor} (p^l - er)^{\phi(p^l)-2} \\ &\equiv -\frac{e^{\phi(p^l)-2+2}}{\phi(p^l) - 1} B_{\phi(p^l)-1}\left(\frac{s}{e}\right) \\ &\equiv -\frac{B_{\phi(p^l)-1}\left(\frac{s}{e}\right)}{\phi(p^l) - 1} \equiv -J_e(p^l) \frac{B_{\phi(p^l)-1}\left(\frac{1}{e}\right)}{\phi(p^l) - 1} \pmod{p^l}, \end{aligned}$$

where  $s$  is the least positive residue of  $p^l$  modulo  $e$ . Hence, (3.1) is valid.

Secondly, we prove that for a positive integer  $m$ , if  $(m, e) = 1$ , then

$$\sum_{\substack{r=1 \\ p \nmid r}}^{\lfloor mp^l/e \rfloor} \frac{1}{r^2} \equiv J_e(m) \sum_{\substack{r=1 \\ p \nmid r}}^{\lfloor p^l/e \rfloor} \frac{1}{r^2}. \quad (3.2)$$

Using Lemma 2.2, we have

if  $m \equiv 1 \pmod{e}$ , then  $m = ek + 1$  for some nonnegative integer  $k$  and

$$\begin{aligned} \sum_{\substack{r=1 \\ p \nmid r}}^{\lfloor (ek+1)p^l/e \rfloor} \frac{1}{r^2} &= \sum_{\substack{r=1 \\ p \nmid r}}^{\lfloor kp^l+p^l/e \rfloor} \frac{1}{r^2} = \sum_{\substack{r=1 \\ p \nmid r}}^{kp^l} \frac{1}{r^2} + \sum_{\substack{r=kp^l+1 \\ p \nmid r}}^{kp^l+\lfloor p^l/e \rfloor} \frac{1}{r^2} \\ &\equiv \sum_{a=0}^{k-1} \sum_{\substack{b=1 \\ p \nmid b}}^{p^l} \frac{1}{(ap^l + b)^2} + \sum_{\substack{r=1 \\ p \nmid r}}^{\lfloor p^l/e \rfloor} \frac{1}{(kp^l + r)^2} \\ &\equiv \sum_{a=0}^{k-1} \sum_{\substack{b=1 \\ p \nmid b}}^{p^l} \frac{1}{b^2} + \sum_{\substack{r=1 \\ p \nmid r}}^{\lfloor p^l/e \rfloor} \frac{1}{r^2} \\ &\equiv k \sum_{\substack{b=1 \\ p \nmid b}}^{p^l-1} \frac{1}{b^2} + \sum_{\substack{r=1 \\ p \nmid r}}^{\lfloor p^l/e \rfloor} \frac{1}{r^2} \\ &\equiv \sum_{\substack{r=1 \\ p \nmid r}}^{\lfloor p^l/e \rfloor} \frac{1}{r^2} \pmod{p^l}; \end{aligned}$$

if  $m \equiv -1 \pmod{e}$ , then  $m = ek - 1$  for some positive integer  $k$  and

$$\sum_{\substack{r=1 \\ p \nmid r}}^{\lfloor (ek-1)p^l/e \rfloor} \frac{1}{r^2} = \sum_{\substack{r=1 \\ p \nmid r}}^{\lfloor (k-1)p^l+(e-1)p^l/e \rfloor} \frac{1}{r^2} \equiv \sum_{\substack{r=1 \\ p \nmid r}}^{\lfloor (e-1)p^l/e \rfloor} \frac{1}{r^2}$$

$$\begin{aligned}
&\equiv \sum_{\substack{r=1 \\ p \nmid r}}^{p^l - \lfloor p^l/e \rfloor - 1} \frac{1}{r^2} \equiv \sum_{\substack{r=1 \\ p \nmid r}}^{p^l-1} \frac{1}{r^2} - \sum_{\substack{r=1 \\ p \nmid r}}^{\lfloor p^l/e \rfloor} \frac{1}{r^2} \\
&\equiv - \sum_{\substack{r=1 \\ p \nmid r}}^{\lfloor p^l/e \rfloor} \frac{1}{r^2} \pmod{p^l}.
\end{aligned}$$

So (3.2) is valid. Furthermore, if  $p^l \parallel n$ , taking  $m = \frac{n}{p^l}$  into (3.2) yields that

$$\sum_{\substack{r=1 \\ p \nmid r}}^{\lfloor n/e \rfloor} \frac{1}{r^2} \equiv -J_e(n) \frac{B_{\phi(p^l)-1}(\frac{1}{e})}{\phi(p^l) - 1} \pmod{p^l}. \quad (3.3)$$

Thirdly, we prove that

$$\sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \frac{1}{r^2} \equiv -J_e(n) n^{\phi(n)-2} \phi_{J_e}^{(2-\phi(n))}(n) \frac{B_{\phi(p^l)-1}(\frac{1}{e})}{\phi(p^l) - 1} \pmod{p^l}, \quad (3.4)$$

Assume  $p_1, p_2, \dots, p_u$  are different prime factors of  $n$ . By noticing that  $\phi(n) - 1 \geq \phi(p^l) - 1 = p^{l-1}(p-1) - 1 \geq 4 \cdot 5^{l-1} - 1 > l$ , we have

$$\begin{aligned}
\sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \frac{1}{r^2} &= \sum_{\substack{r=1 \\ p \nmid r}}^{\lfloor n/e \rfloor} \frac{1}{r^2} - \sum_i \sum_{\substack{r=1 \\ p \nmid r \\ p_i \mid r}}^{\lfloor n/e \rfloor} \frac{1}{r^2} + \sum_{i,j} \sum_{\substack{r=1 \\ p \nmid r \\ p_i p_j \mid r}}^{\lfloor n/e \rfloor} \frac{1}{r^2} + \dots + (-1)^u \sum_{\substack{r=1 \\ p \nmid r \\ p_1 p_2 \dots p_u \mid r}}^{\lfloor n/e \rfloor} \frac{1}{r^2} \\
&\equiv - \left\{ J_e(n) - \sum_i \frac{J_e(n/p_i)}{p_i^2} + \sum_{i,j} \frac{J_e(n/p_i p_j)}{p_i^2 p_j^2} + \dots \right. \\
&\quad \left. + (-1)^u \frac{J_e(n/p_i p_j \dots p_u)}{p_i^2 p_j^2 \dots p_u^2} \right\} \frac{B_{\phi(p^l)-1}(\frac{1}{e})}{\phi(p^l) - 1} \\
&\equiv - \left\{ J_e(n) - \sum_i \frac{J_e(n)}{p_i^2 J_e(p_i)} + \sum_{i,j} \frac{J_e(n)}{p_i^2 p_j^2 J_e(p_i p_j)} + \dots \right. \\
&\quad \left. + (-1)^u \frac{J_e(n)}{p_i^2 p_j^2 \dots p_u^2 J_e(p_i p_j \dots p_u)} \right\} \frac{B_{\phi(p^l)-1}(\frac{1}{e})}{\phi(p^l) - 1} \\
&\equiv -J_e(n) \prod_{q \mid \frac{n}{p^l}} \left( 1 - \frac{1}{q^2 J_e(q)} \right) \frac{B_{\phi(p^l)-1}(\frac{1}{e})}{\phi(p^l) - 1} \\
&\equiv -J_e(n) \prod_{q \mid n} (1 - q^{\phi(p^l)-2} J_e(q)) \frac{B_{\phi(p^l)-1}(\frac{1}{e})}{\phi(p^l) - 1}
\end{aligned}$$

$$\equiv -J_e(n)n^{\phi(n)-2}\phi_{J_e}^{(2-\phi(n))}(n)\frac{B_{\phi(p^l)-1}(\frac{1}{e})}{\phi(p^l)-1} \pmod{p^l},$$

which means (3.4) is valid.

Finally, taking  $k = \phi(p^l)$ ,  $m = e$ ,  $x = 0$ ,  $a = 1$  and  $q = p^l$  in (2.3), we have

$$\begin{aligned} \frac{B_{\phi(p^l)-1}(\frac{1}{e})}{\phi(p^l)-1} &\equiv e \sum_{j=0}^{p^l-1} (\lfloor \frac{1+je}{p^l} \rfloor + \frac{1-e}{2})(1+je)^{\phi(p^l)-2} \pmod{p^l} \\ &\equiv e \sum_{\substack{j=0 \\ (p,1+je)=1}}^{p^l-1} (\lfloor \frac{1+je}{p^l} \rfloor + \frac{1-e}{2})(1+je)^{-2} \pmod{p^l}. \end{aligned}$$

Changing  $k$  to  $\phi(n) - 1$ , we have

$$\begin{aligned} \frac{B_{\phi(n)-1}(\frac{1}{e})}{\phi(n)-1} &\equiv e \sum_{j=0}^{p^l-1} (\lfloor \frac{1+j}{p^l} \rfloor + \frac{1-e}{2})(1+je)^{\phi(n)-2} \pmod{p^l} \\ &\equiv e \sum_{\substack{j=0 \\ (p,1+je)=1}}^{p^l-1} (\lfloor \frac{1+je}{p^l} \rfloor + \frac{1-e}{2})(1+je)^{-2} \pmod{p^l}, \end{aligned}$$

which means for  $p^l \parallel n$ ,

$$\frac{B_{\phi(n)-1}(\frac{1}{e})}{\phi(n)-1} \equiv \frac{B_{\phi(p^l)-1}(\frac{1}{e})}{\phi(p^l)-1} \pmod{p^l}. \quad (3.5)$$

Hence, we can obtain (1.4) □

*Proof of Corollary 1.3.* It follows by Theorem 1.1 and Lemma 2.5. Since

$$\begin{aligned} \sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \frac{1}{n-er} &\equiv \sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} (n-er)^{\phi(n^2)-1} \\ &\equiv \sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \left\{ (-er)^{\phi(n^2)-1} + (\phi(n^2)-1)n(-er)^{\phi(n^2)-2} \right\} \\ &\equiv - \sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} (er)^{\phi(n^2)-1} + \sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} (n\phi(n)-1)n(er)^{\phi(n^2)-2} \\ &\equiv -\frac{1}{e} \sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \frac{1}{r} - \frac{n}{e^2} \sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \frac{1}{r^2} \pmod{n^2}, \end{aligned}$$



we have

$$\sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \frac{1}{r} \equiv -e \sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \frac{1}{n - er} - \frac{n}{e} \sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \frac{1}{r^2} \pmod{n^2}. \quad (3.6)$$

Using Theorem 1.1 and Lemma 2.5, we can obtain the result.  $\square$

*Proof of Theorem 1.2 and Theorem 1.3.* For any positive integer  $e$ , we have

$$\binom{kn-1}{\lfloor n/e \rfloor} = \prod_{r=1}^{\lfloor n/e \rfloor} \frac{kn-r}{r} = \prod_{d|n} \prod_{\substack{r=1 \\ (r,n)=d}}^{\lfloor n/e \rfloor} \frac{kn-r}{r} = \prod_{d|n} T_{n/d} = \prod_{d|n} T_d.$$

Here

$$T_d = \prod_{\substack{r=1 \\ (r,d)=1}}^{\lfloor d/e \rfloor} \frac{kd-r}{r}. \quad (3.7)$$

Now it follows by the multiplicative version of Möbius inversion formula that

$$T_n = \prod_{d|n} \binom{kd-1}{\lfloor d/e \rfloor}^{\mu(n/d)}. \quad (3.8)$$

As for  $T_n$ , we have

$$\begin{aligned} T_n &= \prod_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \frac{kn-r}{r} = (-1)^{\phi_e(n)} \prod_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \left(1 - \frac{kn}{r}\right) \\ &\equiv (-1)^{\phi_e(n)} \left\{ 1 - kn \sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \frac{1}{r} \right. \\ &\quad \left. + \frac{k^2 n^2}{2} \left( \left( \sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \frac{1}{r} \right)^2 - \sum_{\substack{r=1 \\ (r,n)=1}}^{\lfloor n/e \rfloor} \frac{1}{r^2} \right) \right\} \pmod{n^3}. \end{aligned}$$

If  $e = 2$ , then  $\lfloor n/2 \rfloor = \frac{n-1}{2}$ . Note that

$$\sum_{\substack{r=1 \\ (r,n)=1}}^{(n-1)/2} \frac{1}{r^2} = \frac{1}{2} \sum_{\substack{r=1 \\ (r,n)=1}}^{(n-1)/2} \left\{ \frac{1}{r^2} + \frac{1}{(n-r)^2} \right\} = \frac{1}{2} \sum_{\substack{r=1 \\ (r,n)=1}}^{n-1} \frac{1}{r^2} \pmod{n}.$$

Now by Lemmas 2.2 and 2.4, if  $3 \nmid n$  we have

$$\begin{aligned} T_n &\equiv (-1)^{\phi(n)/2} \{1 + 2knq_2(n) + (2k^2 - k)(nq_2(n))^2\} \\ &\equiv (-1)^{\phi(n)/2} (1 + nq_2(n))^{2k} \\ &\equiv (-1)^{\phi(n)/2} 4^{k\phi(n)} \pmod{n^3}. \end{aligned}$$

If  $3 \mid n$ , we replace the moduli by  $n^3/3$ . It follows by combining it with (3.8) that

$$\prod_{d \mid n} \binom{kd-1}{(d-1)/2}^{\mu(n/d)} \equiv (-1)^{\phi(n)/2} 4^{k\phi(n)} \begin{cases} (\text{mod } n^3) & \text{if } 3 \nmid n, \\ (\text{mod } n^3/3) & \text{if } 3 \mid n. \end{cases}$$

If  $e = 3$ , by Theorem 1.1 and Corollary 1.3, for  $(n, 6) = 1$ , we have

$$\begin{aligned} T_n &\equiv (-1)^{\phi_3(n)} \left\{ 1 - kn \left( -\frac{3}{2} q_3(n) + \frac{3}{4} n q_3^2(n) \right) - \frac{1}{3} k n^2 A_3(n) + \frac{1}{2} k^2 n^2 \left( \frac{9}{4} q_3^2(n) + A_3(n) \right) \right\} \\ &\equiv (-1)^{\phi_3(n)} \left\{ 1 + \frac{3}{2} k n q_3(n) - \frac{3}{4} k n^2 q_3^2(n) + \frac{9}{8} k^2 n^2 q_3^2(n) + k \left( \frac{1}{2} - \frac{1}{3} \right) n^2 A_3(n) \right\} \\ &\equiv (-1)^{\phi_3(n)} \left\{ \frac{1}{2} ((1 + n q_3(n))^{3k} + 1) + k \left( \frac{1}{2} - \frac{1}{3} \right) n^2 A_3(n) \right\} \\ &\equiv (-1)^{\phi_3(n)} \left\{ \frac{1}{2} (27^{k\phi(n)} + 1) + k \left( \frac{1}{2} k - \frac{1}{3} \right) n^2 A_3(n) \right\} \pmod{n^3} \end{aligned}$$

Similarly, one may deduce (1.14) and (1.15). This completes the proof of Theorem 1.3.  $\square$

*Proof of Corollary 1.5.* It follows by Theorem 1.2 that we only need to deal with the case  $p = 3$  and  $q \geq 5$ . From the proof of Theorem 1.2, we notice that

$$\binom{3kq-1}{(3q-1)/2} \equiv \left\{ 4^{2k(q-1)} + \frac{3^2 k^2 q^2}{4} \sum_{\substack{r=1 \\ (r,3q)=1}}^{3q-1} \frac{1}{r^2} \right\} \binom{3k-1}{1} \binom{kq-1}{(q-1)/2} \pmod{3^3 q^3}.$$

From Lemma 2.2, we have

$$\sum_{\substack{r=1 \\ (r,3q)=1}}^{3q-1} \frac{1}{r^2} \equiv 0 \pmod{q}.$$

It also follows by the Fermat's little theorem that

$$\sum_{\substack{r=1 \\ (r,3q)=1}}^{3q-1} \frac{1}{r^2} \equiv \sum_{\substack{r=1 \\ (r,3q)=1}}^{3q-1} 1 = 2q - 2 \pmod{3}.$$

Now if  $q \equiv 1 \pmod{6}$ , then

$$\sum_{\substack{r=1 \\ (r,3q)=1}}^{3q-1} \frac{1}{r^2} \equiv 0 \pmod{3q}.$$

We therefore obtain (1.12). If  $q \equiv 5 \pmod{6}$ , then

$$\sum_{\substack{r=1 \\ (r,3q)=1}}^{3q-1} \frac{1}{r^2} \equiv 4q \pmod{3q}.$$

To prove Corollary 1.5, it suffice to show

$$3^2 k^2 q^3 \binom{3k-1}{1} \binom{kq-1}{(q-1)/2} \equiv 0 \pmod{3^3 q^3}. \quad (3.9)$$

For a prime  $p$ , let  $\text{ord}_p(n) := \max\{i \in \mathbb{N} : p^i \mid n\}$ . The Legendre theorem tells

$$\text{ord}_p(n!) = \sum_{i \geq 1} \left\lfloor \frac{n}{p^i} \right\rfloor.$$

We therefore have

$$\begin{aligned} \text{ord}_3 \left( \binom{kq-1}{(q-1)/2} \right) &= \sum_{i \geq 1} \left( \left\lfloor \frac{kq-1}{3^i} \right\rfloor - \left\lfloor \frac{(q-1)/2}{3^i} \right\rfloor - \left\lfloor \frac{((2k-1)q-1)/2}{3^i} \right\rfloor \right) \\ &\geq \left\lfloor \frac{kq-1}{3} \right\rfloor - \left\lfloor \frac{q-1}{6} \right\rfloor - \left\lfloor \frac{(2k-1)q-1}{6} \right\rfloor \\ &= 1 + \left\lfloor \frac{5k-1}{3} \right\rfloor - \left\lfloor \frac{5k}{3} \right\rfloor. \end{aligned}$$

If  $3 \mid k$ , it is obvious that (3.9) holds. If  $3 \nmid k$ , then

$$\text{ord}_3 \left( \binom{kq-1}{(q-1)/2} \right) \geq 1,$$

which implies  $3 \mid \binom{kq-1}{(q-1)/2}$ . This leads to (3.9) immediately.  $\square$

The proof of Theorem 1.4 is more complicated than the previous two. We note that

$$\begin{aligned} 2^{(n-1)/2} \binom{(kn-1)/2}{(n-1)/2} &= \prod_{r=1}^{(n-1)/2} \frac{kn - (2r-1)}{r} \\ &= \prod_{r=1}^{(n-1)/2} \left( \frac{kn-r}{r} \cdot \frac{kn-(n-r)}{r} \cdot \frac{r}{kn-2r} \right) \\ &= \prod_{d|n} \prod_{\substack{r=1 \\ (r,n)=d}}^{(n-1)/2} \left( \frac{kn-r}{r} \cdot \frac{kn-(n-r)}{r} \cdot \frac{r}{kn-2r} \right) \\ &= \prod_{d|n} S_{n/d} = \prod_{d|n} S_d. \end{aligned}$$

Here

$$S_d = \prod_{\substack{r=1 \\ (r,d)=1}}^{(d-1)/2} \left( \frac{kd-r}{r} \cdot \frac{kd-(d-r)}{r} \cdot \frac{r}{kd-2r} \right).$$

Again it follows by the multiplicative version of Möbius inversion formula that

$$S_n = \prod_{d|n} \left( 2^{(d-1)/2} \binom{(kd-1)/2}{(d-1)/2} \right)^{\mu(n/d)}$$

$$= 2^{\phi(n)/2} \prod_{d|n} \binom{(kd-1)/2}{(d-1)/2}^{\mu(n/d)}. \quad (3.10)$$

*Proof of Theorem 1.4.* Now we assume  $3 \nmid n$  for convenience. Otherwise, replacing the moduli when needed. In the proof of Theorem 1.2, we have already obtained

$$\prod_{\substack{r=1 \\ (r,n)=1}}^{(n-1)/2} \frac{kn-r}{r} \equiv (-1)^{\phi(n)/2} 4^{k\phi(n)} \pmod{n^3}. \quad (3.11)$$

Next

$$\begin{aligned} \prod_{\substack{r=1 \\ (r,n)=1}}^{(n-1)/2} \frac{kn-(n-r)}{r} &= \prod_{\substack{r=1 \\ (r,n)=1}}^{(n-1)/2} \left( 1 + \frac{(k-1)n}{r} \right) \\ &\equiv 1 + (k-1)n \sum_{\substack{r=1 \\ (r,n)=1}}^{(n-1)/2} \frac{1}{r} + \frac{(k-1)^2 n^2}{2} \left( \left( \sum_{\substack{r=1 \\ (r,n)=1}}^{(n-1)/2} \frac{1}{r} \right)^2 - \sum_{\substack{r=1 \\ (r,n)=1}}^{(n-1)/2} \frac{1}{r^2} \right) \\ &\equiv 1 - 2(k-1)nq_2(n) + (2(k-1)^2 + (k-1))(nq_2(n))^2 \\ &\equiv (1 + nq_2(n))^{-2(k-1)} \\ &\equiv 4^{-(k-1)\phi(n)} \pmod{n^3}. \end{aligned} \quad (3.12)$$

Then

$$\begin{aligned} \prod_{\substack{r=1 \\ (r,n)=1}}^{(n-1)/2} \frac{kn-2r}{r} &= (-1)^{\phi(n)/2} \prod_{\substack{r=1 \\ (r,n)=1}}^{(n-1)/2} \left( 2 - \frac{kn}{r} \right) \\ &\equiv (-1)^{\phi(n)/2} 2^{\phi(n)/2} \left\{ 1 - \frac{kn}{2} \sum_{\substack{r=1 \\ (r,n)=1}}^{(n-1)/2} \frac{1}{r} \right. \\ &\quad \left. + \frac{1}{4} \frac{k^2 n^2}{2} \left( \left( \sum_{\substack{r=1 \\ (r,n)=1}}^{(n-1)/2} \frac{1}{r} \right)^2 - \sum_{\substack{r=1 \\ (r,n)=1}}^{(n-1)/2} \frac{1}{r^2} \right) \right\} \\ &\equiv (-1)^{\phi(n)/2} 2^{\phi(n)/2} \left\{ 1 + knq_2(n) + \frac{k^2 - k}{2} (nq_2(n))^2 \right\} \\ &\equiv (-1)^{\phi(n)/2} 2^{\phi(n)/2} (1 + nq_2(n))^k \\ &\equiv (-1)^{\phi(n)/2} 2^{\phi(n)/2} 2^{k\phi(n)} \pmod{n^3}. \end{aligned} \quad (3.13)$$

It follows by (3.11), (3.12) and (3.13) that

$$S_n = \prod_{\substack{r=1 \\ (r,n)=1}}^{(n-1)/2} \left( \frac{kn-r}{r} \cdot \frac{kn-(n-r)}{r} \cdot \frac{r}{kn-2r} \right) \equiv 2^{-(k-3/2)\phi(n)} \pmod{n^3}.$$

Combining it with (3.10) we therefore have

$$\prod_{d|n} \left( \frac{(kd-1)/2}{(d-1)/2} \right)^{\mu(n/d)} \equiv 2^{-(k-1)\phi(n)} \begin{cases} \pmod{n^3} & \text{if } 3 \nmid n, \\ \pmod{n^3/3} & \text{if } 3 \mid n. \end{cases}$$

□

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(H. ZHONG) SCHOOL OF MATHEMATICAL SCIENCES, ZHEJIANG UNIVERSITY, HANGZHOU, 310027, CHINA

*E-mail address:* 11435011@zju.edu.cn

(S. CHERN) SCHOOL OF MATHEMATICAL SCIENCES, ZHEJIANG UNIVERSITY, HANGZHOU, 310027, CHINA

*E-mail address:* chenxiaohang92@gmail.com

(T. CAI) SCHOOL OF MATHEMATICAL SCIENCES, ZHEJIANG UNIVERSITY, HANGZHOU, 310027, CHINA

*E-mail address:* txcai@zju.edu.cn